Licenciatura em Gestão

Operational Research Chapter 3

2018-2019

ISEG 100 1911-2011 ANOS

100 ANOS A PENSAR NO FUTURO

3. Duality and sensitivity Analysis

3.1 Introduction

3.2 Duality

3.3 Economic Interpretation of Duality. Shadow Prices. Primal-Dual Relations

3.4 Sensitivity Analysis

- Changes in the Right-Hand Sides of the Constraints
- Changes in the Coefficients of the Objective Function

Prototype example

x_1 – no. batches of P1 produced per week (P1=8-foot glass door with aluminum framing)

x_2 – no. batches of P2 produced per week (P2=4×6 foot double-hung wood framed window)

$$Max\ Z = 3x_1 + 5x_2$$

$$s.t. \begin{cases} x_1 & & \leq 4 \\ & 2x_2 & \leq 12 \\ 3x_1 + & 2x_2 & \leq 18 \\ & x_1, x_2 \geq 0 & \end{cases}$$

Duality

PRIMAL	DUAL
Maximize (Max Z)	Minimize (Min W)
1 constraint	1 decision variable
i-th constraint type "≤"	$y_i \geq 0$
i-th constraint type "≥"	$y_i \leq 0$
i-th constraint type "="	y_i free
RHS: $b_i\ (i = 1,\ldots,m)$	OF: $W = b_1 y_1 + \cdots + b_m y_m$
OF: $Z = c_1 x_1 + \cdots + c_n x_n$	RHS: $c_j\ (j = 1,\ldots,n)$
1 decision variable	1 constraint
$x_j \geq 0$	j-th constraint type "≥"
$x_j \leq 0$	j-th constraint type "≤"
x_j free	j-th constraint type "="
Technical coefficients matrix: A	Technical coefficients matrix: A^T

Duality

(P) Min Z ⇨ (D) Max W

primal constraint "≥" ⇨ dual variable ≥ 0;

primal constraint "≤" ⇨ dual variable ≤ 0;

primal variable ≥ 0 ⇨ dual constraint "≤";

primal variable ≤ 0 ⇨ dual constraint "≥".

O *i*-th shadow price (optimal value of y_i) represents the proportion of change in the optimal value of the primal due to an increase in the *i*-th right hand side.

Duality

Properties

Prop 6: The dual of the dual is the primal.

Propriedade 7: Given a pair of dual problems if a feasible solution x' of the maximization problem takes the value Z' and a feasible solution y'of the minimization problem takes the value W', then Z' ≤ W'.

If Z' = W', then x' and y' are optimal solutions of its problems.

Duality

DUAL \ PRIMAL	with FS	Without FS
with FS	Both problems have optimal solutions and $Z^* = W^*$	Primal infeasible Dual unbounded
Without FS	Primal unbounded Dual infeasible	Primal infeasible Dual infeasible

complementary solutions – verify **complementary slack relations**

Problem (P1) Dual of (P1)

i) decision variable ≠ 0 (BV) ⇨ binding constraint (slack variable =0 NBV)

ii) non binding constraint (slack variable is BV) ⇨ decision variable =0 (NBV)\

Duality

Find DOS: $y_i = \Delta Z$ if $\Delta b_i = +1$

Knowing the Primal solution:

- Graphic – find y_i: solve PL with $\Delta b_i = +1$. Compute: $\Delta Z = y_i$.
- Optimal table– y_i = coefficient in the Z row of the i-th slack variable..
- Complementary slack conditions: Identify NBV (=0) in the dual (D), by the corresponding BV in the (P), and solve the system of equations (augmented form) (D).
- Solver –Sensitivity report –“Shadow Price” Column

LP – solution by Solver

	A	B	C	D	E	F	G
2							Prod.Time Available
3		Plant	doors	windows	total		week/hours
4		1	1	0	0	≤	4
5		2	0	2	0	≤	12
6		3	3	2	0	≤	18
7		profit	3	5	0		
8		no. Batches	0	0			

=SUMPRODUCT(C4:D4;C\$8:D\$8)
=SUMPRODUCT(C5:D5;C\$8:D\$8)
=SUMPRODUCT(C6:D6;C\$8:D\$8)
=SUMPRODUCT(C7:D7;C\$8:D\$8)

LP – Solver: Sensitivity Report

Solution of the dual

Microsoft Excel 14.0 Sensitivity Report
Worksheet: [slidesProto.xlsx]Sheet1
Report Created: 24-10-2011 20:03:38

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$8	no. Batches doors	2	0	3	4,5	3
$D$8	no. Batches windows	6	0	5	1E+30	3

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$4	total	2	0	4	1E+30	2
$E$5	total	12	1,5	12	6	6
$E$6	total	18	1	18	6	6

LP – Simplex

Solution of the dual

Optimal tableaux

Slack variables

	Bas		coefficients							
iter	Var	Eq	Z	x_1	x_2	x_3	x_4	x_5	RHS	Op?
1	Z	0	1	0	0	0	$^3/_2$	1	36	YES
	x_3	1	0	0	0	1	0	$-^1/_3$	2	
	x_2	2	0	0	1	0	$^1/_2$	0	6	
	x_1	3	0	1	0	0	$-^1/_3$	$^1/_3$	2	

Prototype 2

x_1 – no. units of TV advertisement (unit=standart block of advertisement P&G)

x_2 – no. units of advertisement in print media

$$Min\ z = x_1 + 2x_2$$

s.t.

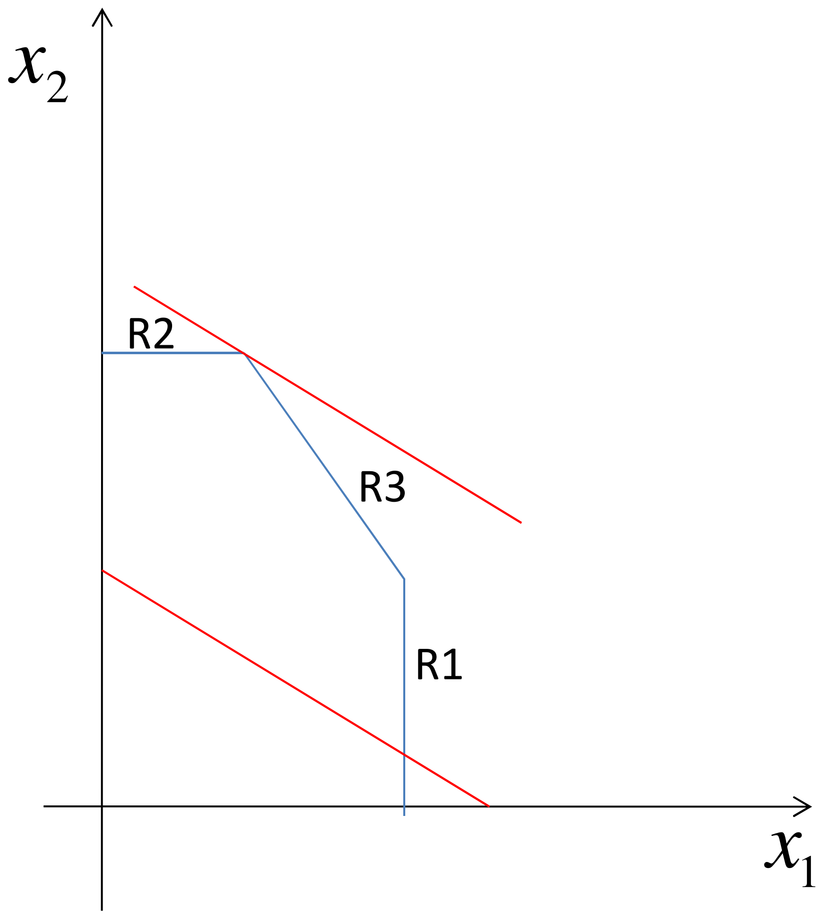
$$\begin{cases} x_2 \geq 3 & \text{(R1)} \\ 3x_1 + 2x_2 \geq 18 & \text{(R2)} \\ -x_1 + 4x_2 \geq 4 & \text{(R3)} \\ x_1, x_2 \geq 0 \end{cases}$$

(R1) min increase sales stain remover

(R2) min increase sales liq. detergent

(R3) min increase sales pow. detergent

Sensitivity Analysis

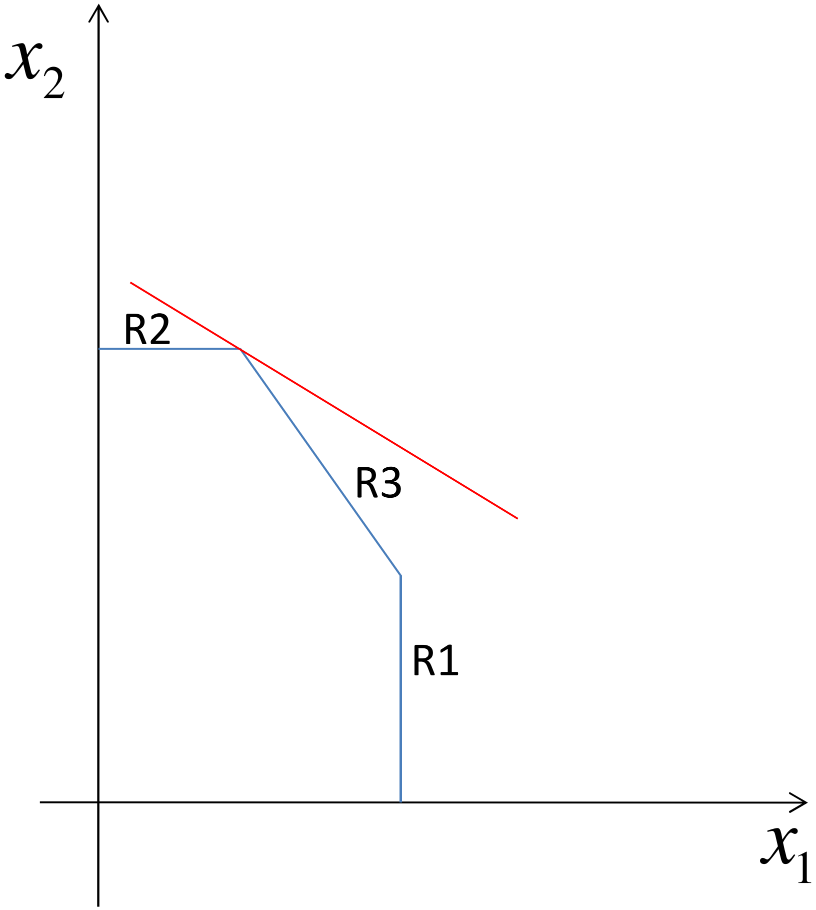


Prototype1

$$Max\ z = 3x_1 + 5x_2$$

$$\begin{cases} x_1 & & \leq & 4 & \text{(R1)} \\ & 2x_2 & \leq & 12 & \text{(R2)} \\ 3x_2 + & 2x_2 & \leq & 18 & \text{(R3)} \\ x_1, & x_2 & \geq & 0 & \end{cases}$$

Sensitivity Analysis



Prototype1

$$Max\ z = 3x_1 + 5x_2$$

$$\begin{cases} x_1 & & \le & 4 & \text{(R1)} \\ & 2x_2 & \le & 12 & \text{(R2)} \\ 3x_2 + & 2x_2 & \le & 18 & \text{(R3)} \\ x_1, & x_2 & \ge & 0 & \end{cases}$$

The optimal solution is achieved in the intersection R2/R3.

$$Z^* = 36; \quad x_1^* = 2; \; x_2^* = 6$$

$$(x_3^* = 2; x_4^* = 0; x_5^* = 0)$$

Sensitivity Analysis

- Is the Optimal solution sensitive to changes in the data, problem parameters?

RHS: b_i

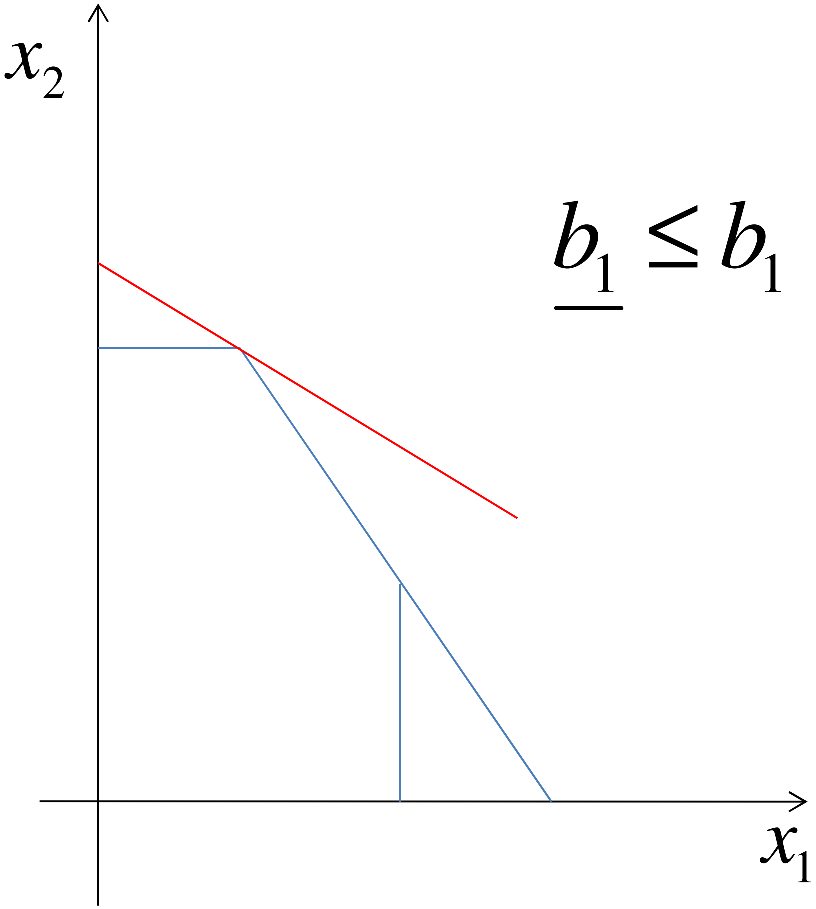
SI – Sensitivity Interval

changes inside SI mantain the optimal sets of basic/ non basc variables

If $\Delta b_i \in \text{SI}$:

- Same optimal base and the shadow prices (DOS)
- Change: POS   and   Z*, $\Delta Z = y_i \times \Delta b_i$

Sensitivity Analysis of RHS



$$\underline{b_1} \leq b_1 \leq \overline{b_1}$$

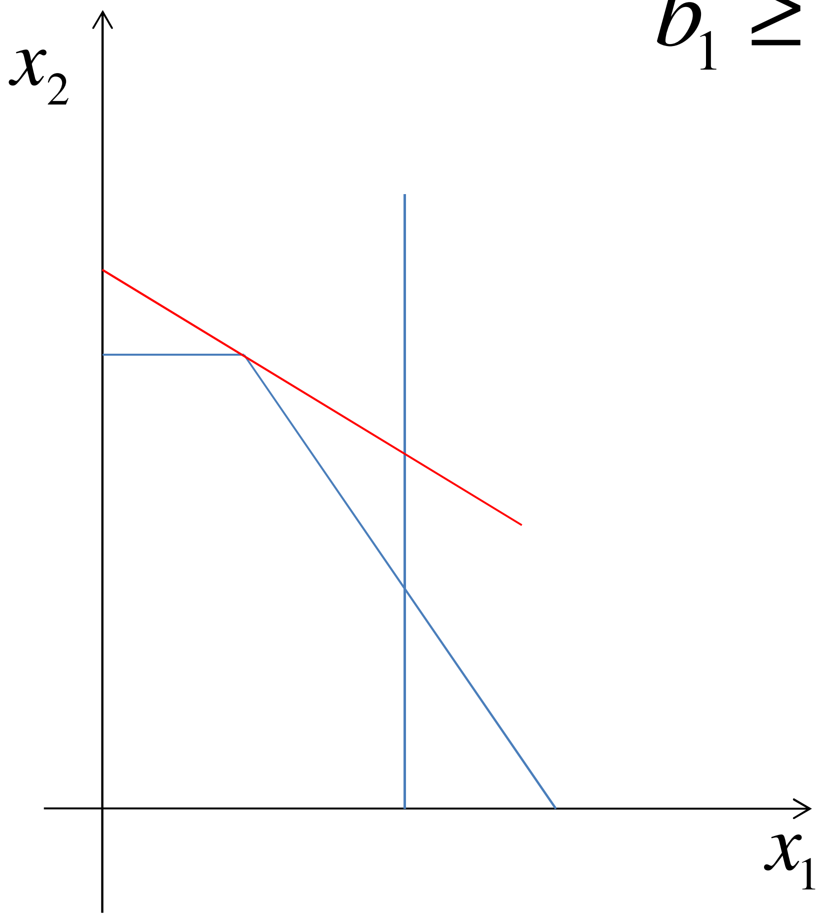
Prototype1

$$Max\ z = 3x_1 + 5x_2$$

$$\begin{cases} x_1 & & \leq & b_1 & \text{(R1)} \\ & 2x_2 & \leq & 12 & \text{(R2)} \\ 3x_2 & + 2x_2 & \leq & 18 & \text{(R3)} \\ x_1, & x_2 & \geq & 0 & \end{cases}$$

$$b_1 \leq \overline{b_1} = \infty$$

Sensitivity Analysis of RHS



$$b_1 \geq 2$$

Prototype1

$$Max\ z = 3x_1 + 5x_2$$

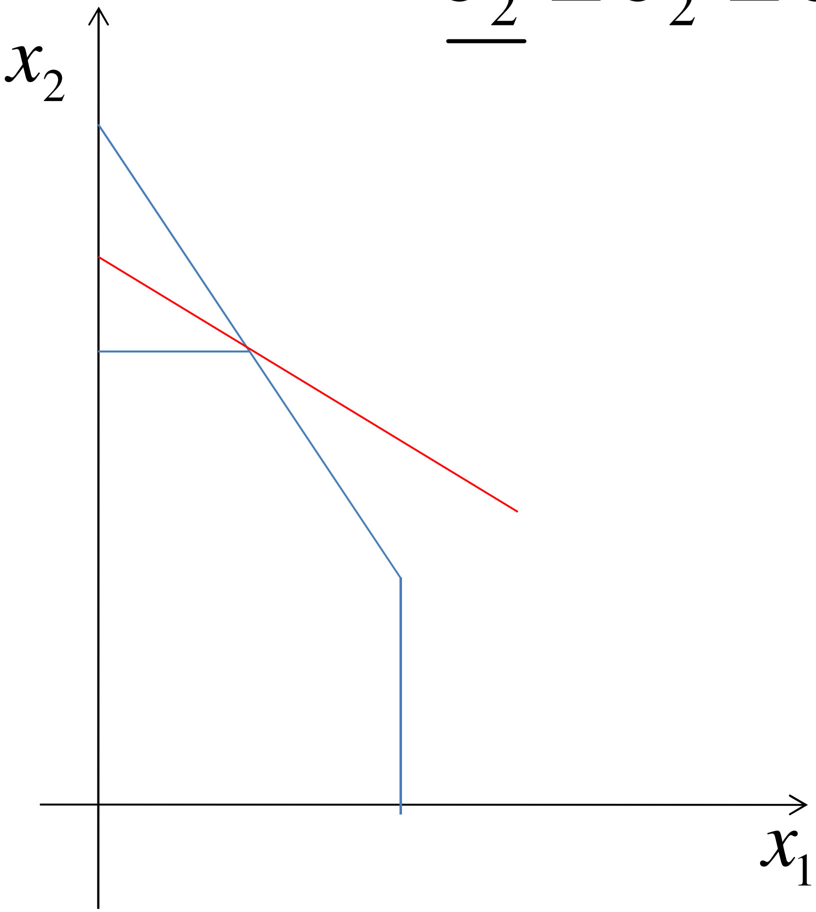
$$\begin{cases} x_1 \leq b_1 & \text{(R1)} \\ 2x_2 \leq 12 & \text{(R2)} \\ 3x_2 + 2x_2 \leq 18 & \text{(R3)} \\ x_1,\ x_2 \geq 0 \end{cases}$$

$$\begin{cases} 2x_2 = 12 \\ x_1 = \underline{b_1} \\ 3x_1 + 2x_2 = 18 \end{cases}$$

$$b_1 \geq \underline{b_1} = 2$$

Sensitivity analysis of RHS

$$\underline{b_2} \le b_2 \le \overline{b_2}$$



Prototype1

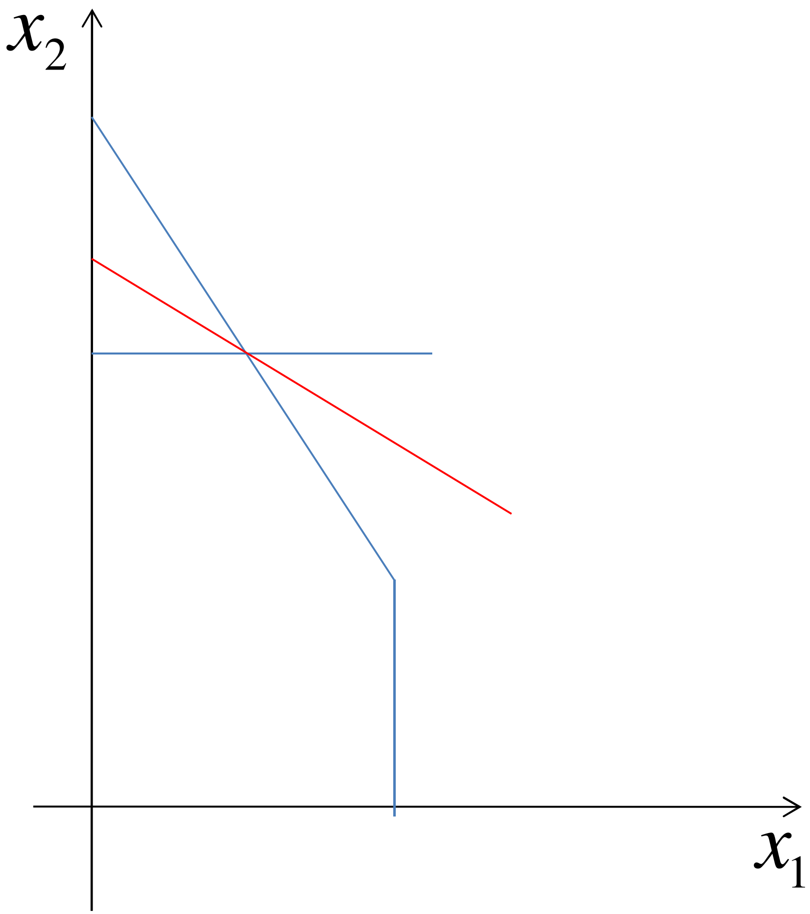
$$Max\ z = 3x_1 + 5x_2$$

$$\begin{cases} x_1 & & \le & 4 & \text{(R1)} \\ & 2x_2 & \le & b_2 & \text{(R2)} \\ 3x_2 + & 2x_2 & \le & 18 & \text{(R3)} \\ x_1, & x_2 & \ge & 0 & \end{cases}$$

$$\begin{cases} & 2x_2 & = & \overline{b_2} \\ x_1 & & = & 0 \\ 3x_1 + & 2x_2 & = & 18 \end{cases}$$

$$b_2 \le \overline{b_2} = 18$$

Sensitivity analysis of RHS



Prototype1

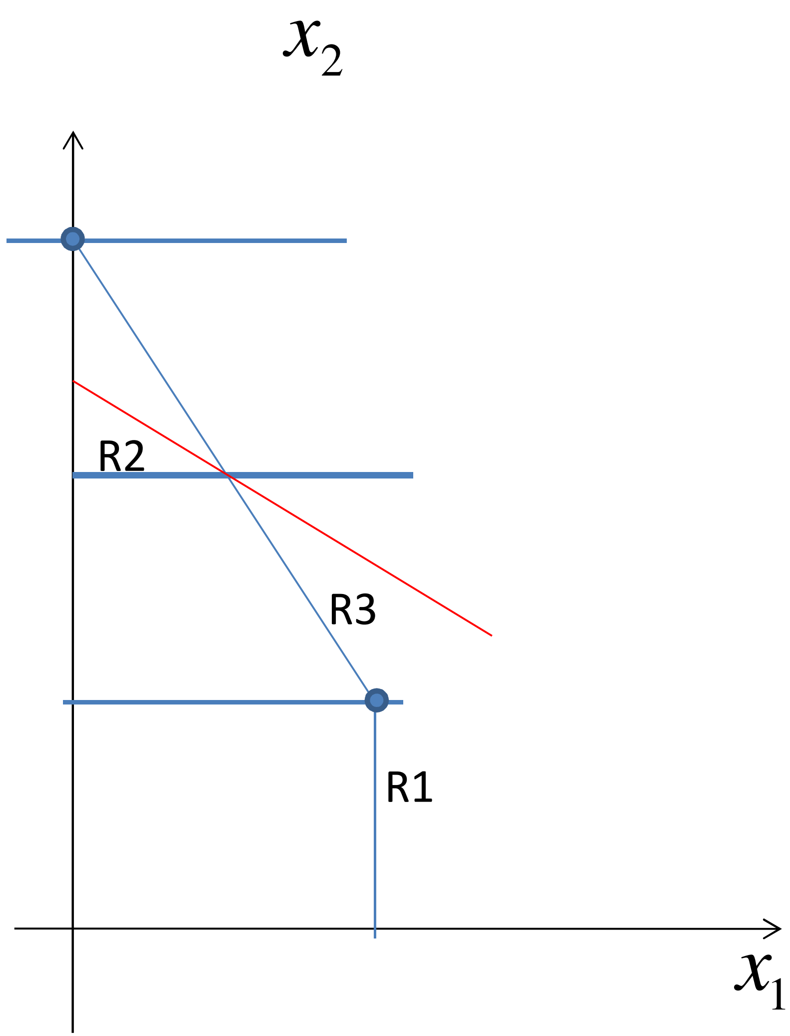
$$Max\ z = 3x_1 + 5x_2$$

$$\begin{cases} x_1 & & \leq & 4 & \text{(R1)} \\ & 2x_2 & \leq & b_2 & \text{(R2)} \\ 3x_2 & + \ 2x_2 & \leq & 18 & \text{(R3)} \\ x_1, & x_2 & \geq & 0 & \end{cases}$$

$$\begin{cases} & 2x_2 & = & \underline{b_2} \\ x_1 & & = & 4 \\ 3x_1 & + \ 2x_2 & = & 18 \end{cases}$$

$$b_2 \geq \underline{b_2} = 6$$

Sensitivity analysis of RHS



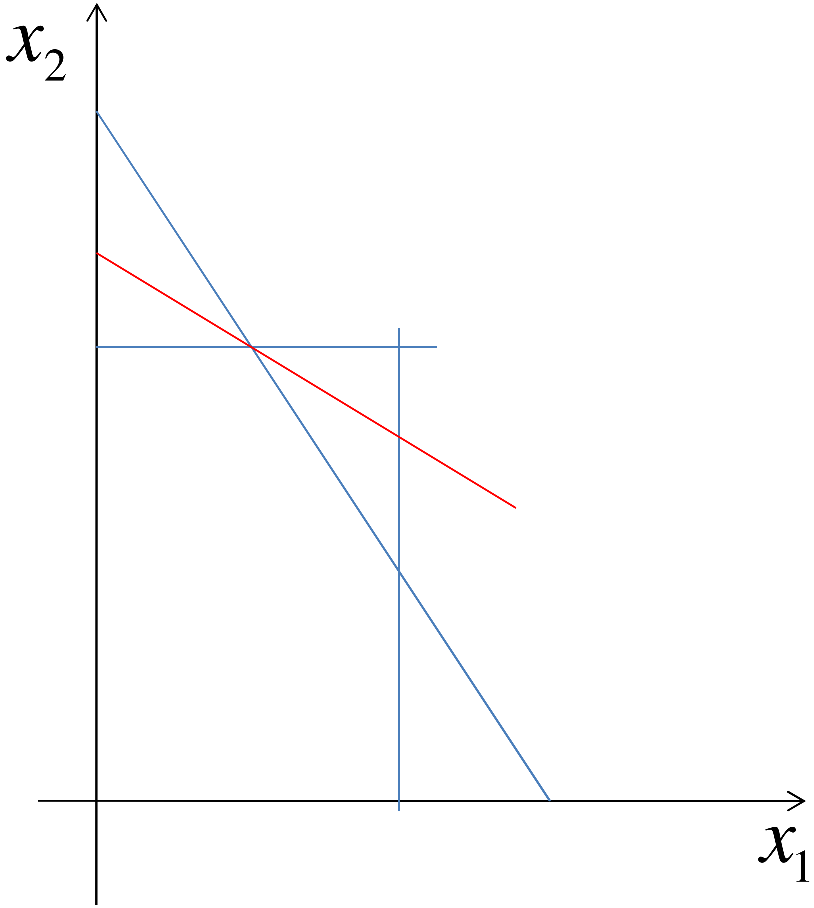
Prototype1

$$Max\ z = 3x_1 + 5x_2$$

$$\begin{cases} x_1 & & \leq & 4 & \text{(R1)} \\ & 2x_2 & \leq & b_2 & \text{(R2)} \\ 3x_2 & + \quad 2x_2 & \leq & 18 & \text{(R3)} \\ x_1, & x_2 & \geq & 0 & \end{cases}$$

If $\underline{6} \leq b_2 \leq 18$
Then the optimal solution still is achieved in the intersection R2/R3

Sensitivity Analysis of RHS



Prototype1

$$Max\ z = 3x_1 + 5x_2$$

$$\begin{cases} x_1 & & \leq & 4 & \text{(R1)} \\ & 2x_2 & \leq & b_2 & \text{(R2)} \\ 3x_2 & + \ 2x_2 & \leq & 18 & \text{(R3)} \\ x_1, & x_2 & \geq & 0 & \end{cases}$$

If $12 \leq b_3 \leq 24$

Then the optimal solution still is achieved in the intersection R2/R3

Sensitivity Analysis of RHS

PROTOTIPO1.xlsx

Microsoft Excel 12.0 Sensitivity Report
Worksheet: [PROTOTIPO1.xlsx]Sheet1
Report Created: 09-11-2010 8:51:23

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$7	solution x1	2	0	3	4,5	3
$D$7	solution x2	6	0	5	1E+30	3

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$3	R1	2	0	4	1E+30	2
$E$4	R2	12	1,5	12	6	6
$E$5	R3	18	1	18	6	6

Answer Report 1 | Sensitivity Report 1 | Limits Report 1 | Sheet1 | Sheet2 | Sheet3

$$b_1 \geq 2$$
$$6 \leq b_2 \leq 18$$
$$12 \leq b_3 \leq 24$$

$$b_1 \geq 4-2$$
$$12-6 \leq b_2 \leq 12+6$$
$$18-6 \leq b_3 \leq 18+6$$

Prototype 2

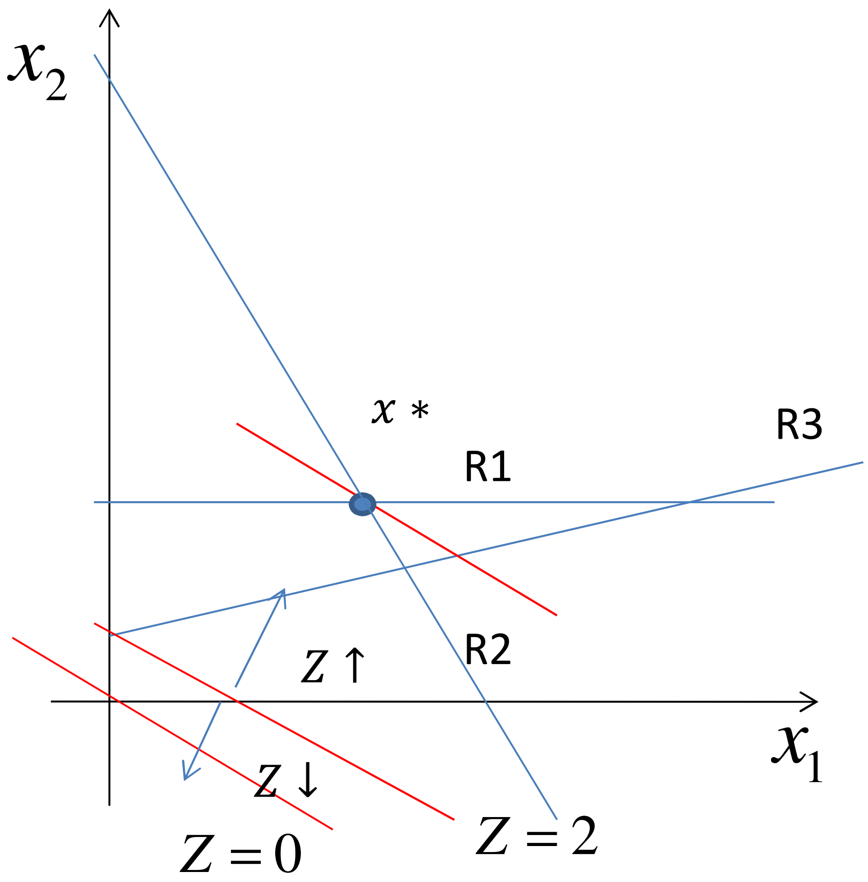
$x_1 -$ no. units of TV advertisement (unit=standart block of advertisement P&G)

$x_2 -$ no. units of advertisement in print media

$$Min\ z = x_1 + 2x_2$$

$$\text{s.t.} \begin{cases} x_2 \geq 3 & \text{(R1)} \\ 3x_1 + 2x_2 \geq 18 & \text{(R2)} \\ -x_1 + 4x_2 \geq 4 & \text{(R3)} \\ x_1, \ x_2 \geq 0 \end{cases}$$

Prototype 2



$$Min\ z = x_1 + 2x_2$$

$$\begin{cases} x_2 \geq 3 & \text{(R1)} \\ 3x_1 + 2x_2 \geq 18 & \text{(R2)} \\ -x_1 + 4x_2 \geq 4 & \text{(R3)} \\ x_1, \ x_2 \geq 0 \end{cases}$$

$$\begin{cases} x_2 = 3 \\ 3x_1 + 2x_2 = 18 \end{cases} \cdots \begin{cases} x^*_1 = 4 \\ x^*_2 = 3 \end{cases}$$

$$Z^* = 10$$

Sensitivity Analysis

- Is the Optimal solution sensitive to changes in the data, problem parameters?

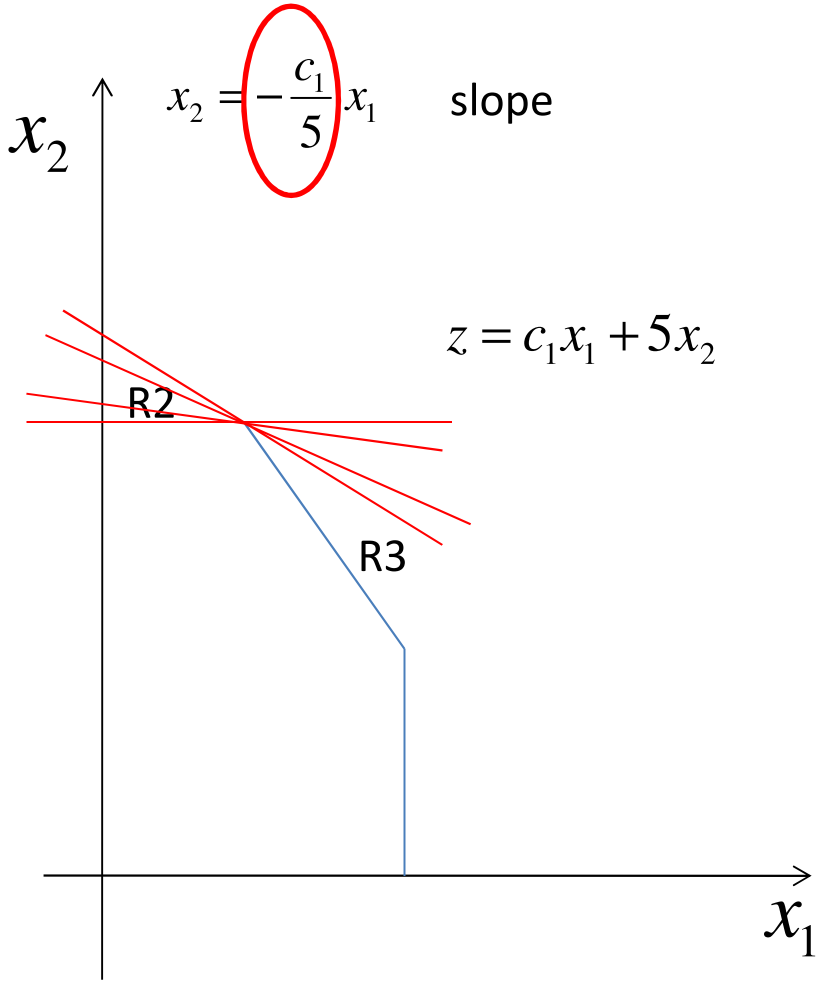
OF coeffficient : c_j
SI – Sensitivity Interval
changes inside SI mantain the optimal sets of basic/ non basc variables

If $\Delta c_j \in \mathrm{SI}$:

- Same optimal base and the primal solution (POS)

- Change: DOS and Z*, $\Delta Z = x_j \times \Delta c_j$

Sensitivity Analysis of OF coeff.



Prototype1

$$
\begin{aligned}
&Max\ z = c_1 x_1 + 5x_2 \\
&\begin{cases}
x_1 & \leq 4 \quad \text{(R1)} \\
2x_2 & \leq 12 \quad \text{(R2)} \\
3x_2 + 2x_2 & \leq 18 \quad \text{(R3)} \\
x_1,\ x_2 & \geq 0
\end{cases}
\end{aligned}
$$

The optimal solution is achieved in the intersection R2/R3.

$$Z^* = 36; \quad x_1^* = 2; x_2^* = 6$$

$$(x_3^* = 2; x_4^* = 0; x_5^* = 0)$$

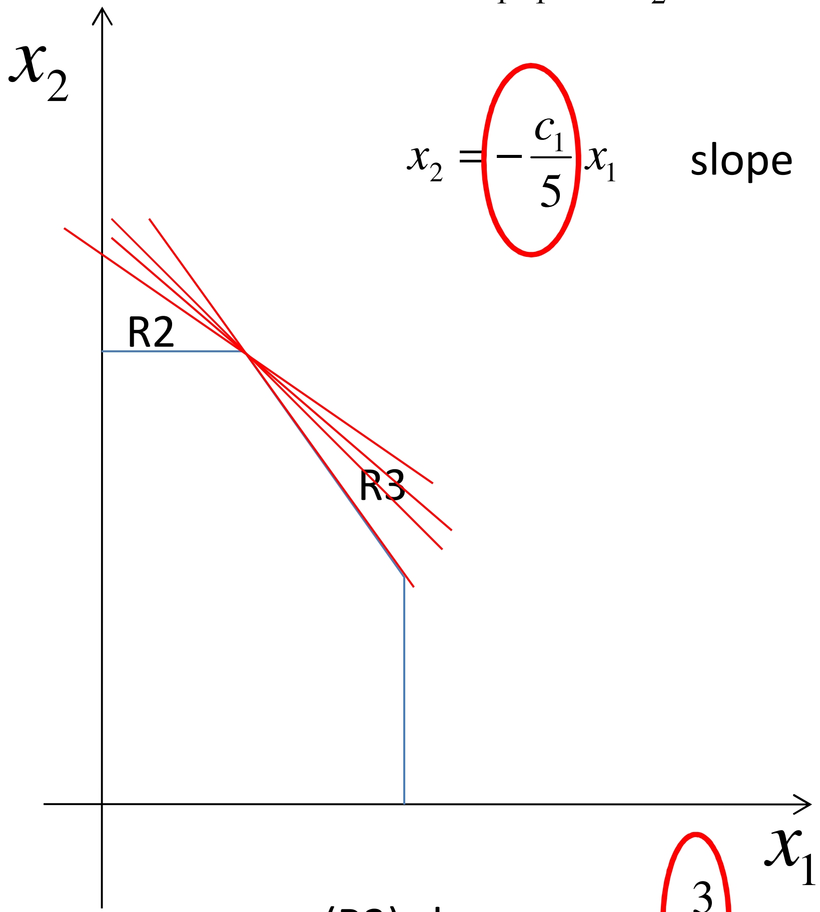
$$-\frac{c_1}{5} \leq 0 \Leftrightarrow c_1 \geq 0$$

(R2) slope $\quad x_2 = 0\, x_1$

Sensitivity Analysis of OF coeff.

$$z = c_1 x_1 + 5x_2$$

$$x_2 = -\frac{c_1}{5} x_1$$ slope



(R3) slope $x_2 = -\frac{3}{2} x_1$

Prototype1

$$Max\ z = c_1 x_1 + 5x_2$$

$$\begin{cases} x_1 \leq 4 & \text{(R1)} \\ 2x_2 \leq 12 & \text{(R2)} \\ 3x_2 + 2x_2 \leq 18 & \text{(R3)} \\ x_1, \quad x_2 \geq 0 \end{cases}$$

The optimal solution is achieved in the intersection R2/R3.

$$Z^* = 36; \quad x_1^* = 2; x_2^* = 6$$

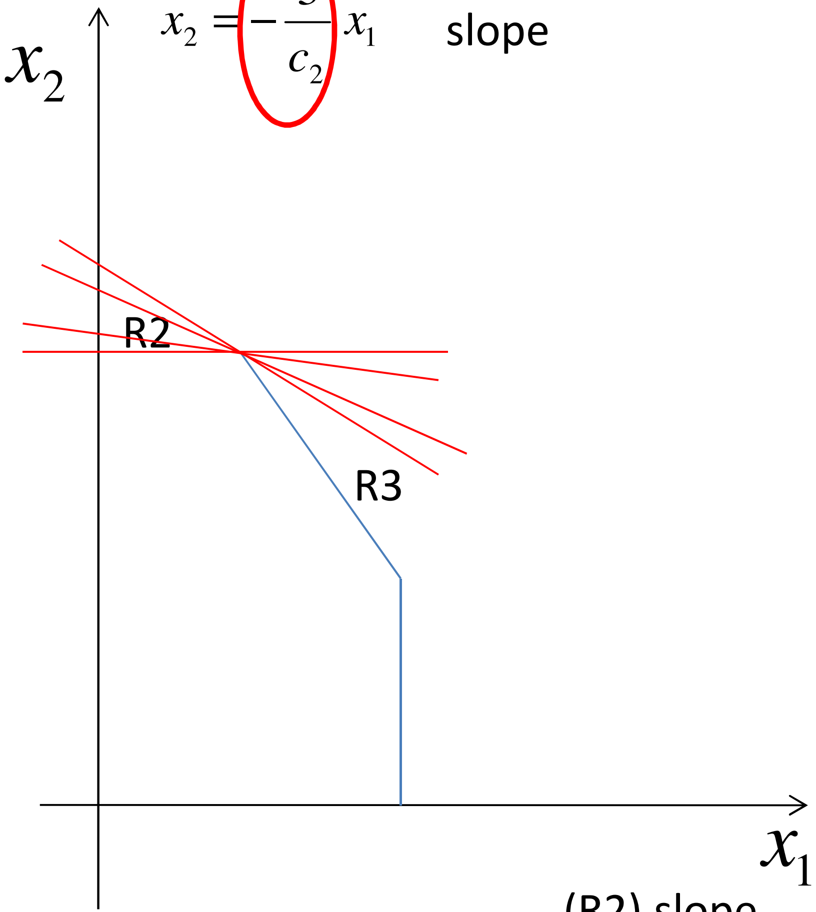
$$(x_3^* = 2; x_4^* = 0; x_5^* = 0)$$

$$-\frac{3}{2} \leq -\frac{c_1}{5} \Leftrightarrow 15 \geq 2c_1 \Leftrightarrow c_1 \leq 7.5$$

Sensitivity Analysis of OF coeff.

$$z = 3x_1 + c_2 x_2$$

$$x_2 = -\frac{3}{c_2} x_1$$ slope



(R2) slope $x_2 = 0\, x_1$

Prototype1

$$\text{Max } z = 3x_1 + c_2 x_2$$

$$\begin{cases} x_1 & & \leq & 4 & \text{(R1)} \\ & 2x_2 & \leq & 12 & \text{(R2)} \\ 3x_2 + & 2x_2 & \leq & 18 & \text{(R3)} \\ x_1, & x_2 & \geq & 0 & \end{cases}$$

The optimal solution is achieved in the intersection R2/R3.

$$Z^* = 36; \quad x_1^* = 2; x_2^* = 6$$

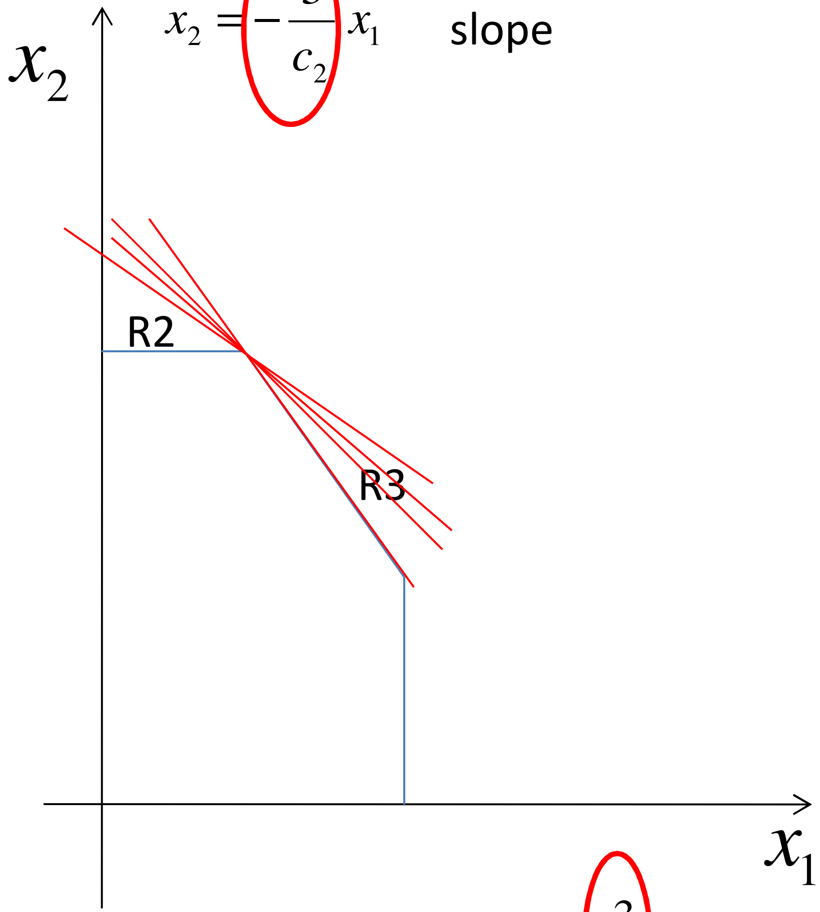
$$(x_3^* = 2; x_4^* = 0; x_5^* = 0)$$

$$-\frac{3}{c_2} \leq 0 \quad \Leftrightarrow \quad c_2 \geq 0$$

Sensitivity Analysis of OF coeff.

$$z = 3x_1 + c_2 x_2$$

$$x_2 = -\frac{3}{c_2} x_1$$ slope



(R3) slope $x_2 = -\frac{3}{2} x_1$

Prototype1

$$Max\ z = 3x_1 + c_2 x_2$$

$$\begin{cases} x_1 & & \leq & 4 & \text{(R1)} \\ & 2x_2 & \leq & 12 & \text{(R2)} \\ 3x_2 & + \ 2x_2 & \leq & 18 & \text{(R3)} \\ x_1, & x_2 & \geq & 0 & \end{cases}$$

The optimal solution is achieved in the intersection R2/R3.

$$Z^* = 36; \quad x_1^* = 2; \ x_2^* = 6$$

$$(x_3^* = 2; x_4^* = 0; x_5^* = 0)$$

$$-\frac{3}{2} \leq -\frac{3}{c_2} \Leftrightarrow -3c_2 \leq -6 \Leftrightarrow \ c_2 \geq 2$$

Sensitivity Analysis of OF coeff.

PROTOTIPO1.xlsx

Microsoft Excel 12.0 Sensitivity Report
Worksheet: [PROTOTIPO1.xlsx]Sheet1
Report Created: 09-11-2010 8:51:23

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$7	solution x1	2	0	3	4,5	3
$D$7	solution x2	6	0	5	1E+30	3

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$3	R1	2	0	4	1E+30	2
$E$4	R2	12	1,5	12	6	6
$E$5	R3	18	1	18	6	6

Answer Report 1 | Sensitivity Report 1 | Limits Report 1 | Sheet1 | Sheet2 | Sheet3

$$0 \le c_1 \le 7.5$$
$$c_2 \ge 2$$

$$3-3 \le c_1 \le 3+4.5$$
$$5-3 \le c_2 \le \infty$$

Sensitivity Analysis

Aditional constraint

New variable

Exercise 7

x_j no. Chocolate type j bars to make

	type 1	type 2	type 3			
sugar	1	1	1	50	≤	50
choc	1	3	2	100	≤	100
profit	3	7	5	250		
	25	25	0			

Microsoft Excel 14.0 Answer Report					
Cell	Name	Original Value	Final Value		
$G$6	profit	0	250		
Cell	Name	Original Value	Final Value	Integer	
$D$7	type 1	0	25	Contin	
$E$7	type 2	0	25	Contin	
$F$7	type 3	0	0	Contin	
Cell	Name	Cell Value	Formula	Status	Slack
$G$4	sugar	50	$G$4<=$I$4	Binding	0
$G$5	choc	100	$G$5<=$I$5	Binding	0

Microsoft Excel 14.0 Sensitivity Report						
Variable Cells						
			Final	Reduced	Objective	Allowable	Allowable
	Cell	Name	Value	Cost	Coefficient	Increase	Decrease
	$D$7	type 1	25	0	3	4	0
	$E$7	type 2	25	0	7	2	0
	$F$7	type 3	0	0	5	0	1E+30
Constraints							
			Final	Shadow	Constraint	Allowable	Allowable
	Cell	Name	Value	Price	R.H. Side	Increase	Decrease
	$G$4	sugar	50	1	50	50	16,6666666 7
	$G$5	choc	100	2	100	50	50